

Analytic structure of three-mass triangle coefficients

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ABSTRACT: “Three-mass triangles” are a class of integral functions appearing in one-loop gauge theory amplitudes. We discuss how the complex analytic properties and singularity structures of these amplitudes can be combined with generalised unitarity techniques to produce compact expressions for three-mass triangle coefficients. We present formulae for the $\mathcal{N} = 1$ contributions to the n -point NMHV amplitude.

KEYWORDS: NLO Computations, QCD.

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1. Introduction

A general n -point one-loop amplitude in a massless theory such as QCD can be expanded in terms of integral functions,

$$A_n^{1\text{-loop}} = \sum_{i \in \mathcal{C}} c_i I_4^i + \sum_{j \in \mathcal{D}} d_j I_3^j + \sum_{k \in \mathcal{E}} e_k I_2^k + R, \quad (1.1)$$

where c_i, d_i, e_i and R are rational functions and the I_4, I_3 , and I_2 are scalar box, triangle and bubble functions respectively. The mathematical form of these integral functions depends on whether the momenta flowing into a vertex are null (massless) or not (massive). This decomposition suggests a “divide and conquer” approach to evaluating one-loop amplitudes where different techniques are used to evaluate the different types of coefficient.

In principle, traditional Feynman diagram techniques, combined with reduction strategies can be used to compute the integral coefficients [1, 2]. Considerable progress has been made in implementing this strategy, however the degree of complexity rises rapidly with the number of external legs and the current state of the art is in the computation of five and six-point amplitudes (see for example [3, 4]).

Alternate approaches based on the physical properties of amplitudes have proved competitive or superior, particularly in computing amplitudes with enhanced symmetry, such as those appearing in supersymmetric theories, or for amplitudes with particular helicity

configurations. Two particularly powerful methods have been those based on unitarity and factorisation. The conjectured duality between the perturbation theory of gauge theories and string theory [5] has provided added insight to these approaches, particularly with respect to complex factorisation.

The unitarity method [6, 7], combined with a knowledge of a basis set of integral functions for an amplitude, provides a systematic way of calculating loop amplitudes. Two-particle cuts provide sufficient information to identify many of the coefficients in eq. (1.1) particularly in cases where the amplitude is “cut-constructible” [6–13]. In addition, extending to D -dimensional unitarity, in principle, provides the information to calculate the rational parts [14]. Three and four particle cuts may also be used to identify coefficients of triangle and box functions [8, 15, 16]. For the box coefficients, c_i , quadruple cuts [15] are particularly simple since the loop momentum integration is frozen by the insertion of four δ -functions.

The analytic structure of the cut integrals appearing in the unitarity method can also be exploited to obtain coefficients. For example, the “holomorphic anomaly” provided an insight into the differing analytic properties of various integral functions in the two-particle cuts [17–19]. Various techniques have been developed to identify the integral coefficients based on the analytic structure of the integrand of the cut [9, 10, 20].

In this paper we explore some recent suggestions for evaluating the coefficients of the “three-mass triangle” integral functions $I_3^m(K_1, K_2, K_3)$ ($K_i^2 \neq 0$) by using the analytic structure of the triple cut. In ref. [21] an algebraic technique was presented for obtaining these coefficients. In this paper we review and refine this technique and present a version that uses a single contour integration.

Although we can divide the amplitude into separate coefficients, in general, different integral coefficients are related by a rich web of “spurious singularities”. These are singularities that are not present in the full amplitude but which appear in individual coefficients. This structure is particularly rich for the three-mass triangles. We explore and use this to obtain compact expressions for these coefficients in the $\mathcal{N} = 1$ contributions to six gluon scattering. These provide alternate forms to those originally calculated in [9]. We present formulae for the n -point “next-to-MHV” (NMHV) $\mathcal{N} = 1$ contribution and describe how results may be obtained in the $N = 0$ and beyond-NMHV cases.

2. Organisation of amplitudes

For one-loop amplitudes with external gluons in QCD it is convenient to decompose the contribution from gluons circulating in the loop into pieces corresponding to complex scalars or supersymmetric multiplets circulating in the loop,

$$A_n^{1\text{-loop}} = A_n^{\mathcal{N}=4} - 4A_n^{\mathcal{N}=1 \text{ chiral}} + A_n^{\text{scalar}}. \quad (2.1)$$

The $A_n^{\mathcal{N}=4}$ component consists entirely of box integrals. The terms we look at in this paper are the $A_n^{\mathcal{N}=1 \text{ chiral}}$ contributions. These amplitudes may contain any of the integral functions (but not rational terms). Furthermore we consider the amplitude to be colour

ordered and focus our attention on the leading in colour component from which the full amplitude can be obtained [6, 22].

The integrals appearing in the amplitude may be box, triangle or bubble functions. We are interested in the contributions of three-mass triangles. The relevant integral function is defined by,

$$I_3^{3m} = i(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{1}{p^2 (p - K_1)^2 (p + K_2)^2}, \quad (2.2)$$

and can be written as [23, 24],

$$I_3^{3m} = \frac{i}{\sqrt{\Delta_3}} \sum_{j=1}^3 \left[\text{Li}_2 \left(- \left(\frac{1 + i\delta_j}{1 - i\delta_j} \right) \right) - \text{Li}_2 \left(- \left(\frac{1 - i\delta_j}{1 + i\delta_j} \right) \right) \right] + \mathcal{O}(\epsilon), \quad (2.3)$$

where,

$$\begin{aligned} \delta_1 &= \frac{K_1^2 - K_2^2 - K_3^2}{\sqrt{\Delta_3}}, \\ \delta_2 &= \frac{K_2^2 - K_1^2 - K_3^2}{\sqrt{\Delta_3}}, \\ \delta_3 &= \frac{K_3^2 - K_1^2 - K_2^2}{\sqrt{\Delta_3}}, \end{aligned} \quad (2.4)$$

and

$$\Delta_3 \equiv -(K_1^2)^2 - (K_2^2)^2 - (K_3^2)^2 + 2K_1^2 K_2^2 + 2K_1^2 K_3^2 + 2K_2^2 K_3^2. \quad (2.5)$$

The other integral functions we will encounter can be obtained in many places e.g. [7]. The one-mass triangles depend only on the momentum invariant of the massive leg, K^2 ,

$$I_3^{1m}(K^2) = \frac{r_\Gamma}{\epsilon^2} (-K^2)^{-1-\epsilon} \equiv G(K^2), \quad (2.6)$$

where $r_\Gamma \equiv \Gamma(1 + \epsilon)\Gamma^2(1 - \epsilon)/\Gamma(1 - 2\epsilon)$. The two-mass triangle integral,

$$I_3^{2m}(K_1^2, K_2^2) = \frac{r_\Gamma}{\epsilon^2} \frac{(-K_1^2)^{-\epsilon} - (-K_2^2)^{-\epsilon}}{(-K_1^2) - (-K_2^2)}, \quad (2.7)$$

can be expressed as one-mass triangle functions,

$$I_3^{2m}(K_1^2, K_2^2) = \frac{1}{(-K_1^2) - (-K_2^2)} (G(K_1^2) - G(K_2^2)), \quad (2.8)$$

and we can drop these functions from our basis of integral functions in favour of $G(K^2)$ functions. The box functions may be found in many places, for example ref. [24, 6]. We need the form of one of these, namely the integral function where two adjacent legs are massless; the so-called ‘‘two-mass hard’’ function. If k_1 and k_2 are the null legs, defining $S \equiv (k_1 + k_2)^2$ and $T = (k_2 + K_3)^2$ we have,

$$\begin{aligned} I_4^{2mh} &= \frac{-2r_\Gamma}{ST} \left\{ -\frac{1}{\epsilon^2} \left[(-S)^{-\epsilon} + (-T)^{-\epsilon} - (-K_3^2)^{-\epsilon} - (-K_4^2)^{-\epsilon} \right] \right. \\ &\quad \left. - \frac{1}{2\epsilon^2} \frac{(-K_3^2)^{-\epsilon} (-K_4^2)^{-\epsilon}}{(-S)^{-\epsilon}} + \frac{1}{2} \ln^2 \left(\frac{S}{T} \right) + \text{Li}_2 \left(1 - \frac{K_3^2}{T} \right) + \text{Li}_2 \left(1 - \frac{K_4^2}{T} \right) \right\}. \end{aligned} \quad (2.9)$$

The coefficients of the integral functions will be expressed as rational functions of spinor inner-products [25], $\langle j l \rangle \equiv \langle k_j^- | k_l^+ \rangle$, $[j l] \equiv \langle k_j^+ | k_l^- \rangle$, where $|k_i^\pm\rangle$ is a massless Weyl spinor with momentum k_i and chirality \pm . We use notation where,

$$\langle k_a^+ | \mathcal{K}_{bcd} | k_e^+ \rangle \equiv [a | K_{bcd} | e \rangle = [a b] \langle b e \rangle + [a c] \langle c e \rangle + [a d] \langle d e \rangle . \quad (2.10)$$

As in twistor-space studies we define,

$$\lambda_i = |k_i^+\rangle , \quad \tilde{\lambda}_i = |k_i^-\rangle . \quad (2.11)$$

3. Singularity structure of six-point three-mass triangles

In this section we look at the three-mass triangle integrals found in six-point one-loop gluon scattering amplitudes. The only non-vanishing three-mass triangle coefficients appear in the NMHV amplitudes, of which there are three inequivalent forms:

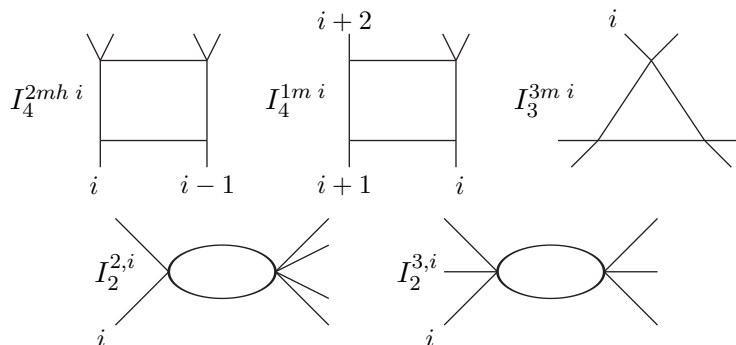
$$A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) , \quad A(1^-, 2^-, 3^+, 4^-, 5^+, 6^+) , \quad A(1^-, 2^+, 3^-, 4^+, 5^-, 6^+) . \quad (3.1)$$

The first of these was calculated in ref. [19] and contains no three-mass triangles, as can be seen from the triple-cuts. The remaining two were computed in ref. [9] using the analytic structure of the two-particle cuts. Although correct (as verified by numerical comparison to a Feynman diagram calculation [3]), these expressions contain irrational expressions involving the square root of the Gram determinant of the three-mass triangle, $\sqrt{\Delta_3}$. We produce expressions with the correct singularity structure which explicitly do not contain these irrational terms.

We start by considering the amplitude $A(1^-, 2^+, 3^-, 4^+, 5^-, 6^+)$. As for any supersymmetric amplitude, the cancellations occurring at one-loop imply that no rational terms appear [7]. Further, by examining the unitarity cuts, we see that only one-mass and two-mass hard boxes appear and, as discussed above, we choose a basis where the two-mass triangles are replaced by one-mass triangle functions, $G(K^2)$. We thus have,

$$\begin{aligned} A^{\mathcal{N}=1}(1^-, 2^+, 3^-, 4^+, 5^-, 6^+) &= \sum_{i=1}^6 c_i^{1m} I_4^{1m i} + \sum_{i=1}^6 c_i^{2mh} I_4^{2mh i} + \sum_{i=1}^6 d_i G(s_{i i+1}) \\ &+ \sum_{i=1}^3 d'_i G(t_{i i+1 i+2}) + d_1^{3m} I_3^{3m 1} + d_2^{3m} I_3^{3m 2} + \sum_{i=1}^6 e_i I_2^{2,i} + \sum_{i=1}^3 e'_i I_2^{3,i} , \end{aligned} \quad (3.2)$$

where both of the three-mass triangles, $I_3^{3m 1}(K_{12}, K_{34}, K_{56})$ and $I_3^{3m 2}(K_{61}, K_{23}, K_{45})$, appear. The labelling of functions is specified below,



We shall see how much of the amplitude can be reconstructed from the singularity structure: both real and spurious. Our starting point is the box coefficients [26],

$$\begin{aligned}
 c_1^{1m} &= i \frac{[2|K|5]^2 [1|K|5] [3|K|5]}{[13]^2 \langle 45 \rangle \langle 56 \rangle [1|K|4] [3|K|6] K^2} \times \frac{-s_{12}s_{23}}{2}, \quad K = K_{123}, \\
 c_4^{2mh} &= i \frac{[2|K|5]^2 [3|K|5] [2|K|4]}{[12] \langle 56 \rangle [3|K|6] [1|K|4] [3|K|4]^2} \times \frac{-s_{34}K^2}{2}, \quad K = K_{123}.
 \end{aligned}
 \tag{3.3}$$

We will see that these completely determine the coefficients of the one-mass triangles and determine much of the three-mass triangle coefficients.

3.1 Infra-red singularities

One of the major constraints on the triangle coefficients comes from requiring that the amplitude has the correct infra-red singularities. The box integral functions with massless legs and the one-mass triangle functions both contain $\ln(K^2)/\epsilon$ singularities. We then have,

$$\left[d_i G(K^2) + \sum c_i I_4^i \right]_{\ln(K^2)/\epsilon} = 0, \tag{3.4}$$

for the $\mathcal{N} = 1$ chiral multiple where such singularities do not appear in the amplitude. These constraints fix the coefficients of the $G(K^2)$ in terms of the box coefficients.

For the $\mathcal{N} = 1$ multiplet, one could choose a basis in which the coefficients of the one-mass triangles are zero. There are several options for doing so: firstly one can choose the basis of “six dimensional scalar box functions” as in [19] or one can choose a basis of functions where the IR singularities have been subtracted as in [9]. In the first case the $D = 4$ boxes and $D = 6$ boxes are related, using the notation of ref. [24],

$$I_4^{D=4} = \frac{1}{2N_4} \left[\sum_i \alpha_i \gamma_i I_3^{(i)} + (-1 + 2\epsilon) \hat{\Delta}_4 I_4^{D=6} \right], \tag{3.5}$$

where $I_3^{(i)}$ is the descendant triangle in which the i -th propagator is deleted. If we change the basis of box integral functions, the coefficients of the triangles, including the three-mass, are shifted,

$$d_i \longrightarrow d_i + \frac{\alpha_i \gamma_i}{2N_4} c_{\text{box}}. \tag{3.6}$$

Although transforming to this basis is instructive, we will continue to use the basis of $D = 4$ integral functions.

3.2 Spurious singularities

Looking at the box coefficients we see that factors such as,

$$\frac{1}{[3|K_{123}|4]^2} \quad \text{and} \quad \frac{1}{[13]^2}, \tag{3.7}$$

appear. The first of these is singular when the momenta are arranged such that,

$$k_1^\mu + k_2^\mu = \alpha k_3^\mu + \beta k_4^\mu, \tag{3.8}$$

and such singularities are termed co-planar.¹ These singularities are *spurious*, meaning they may appear in individual terms within an amplitude but disappear when the entire amplitude is constructed. These coplanar singularities do not cancel amongst the boxes, but cancel between the boxes and the other integral functions. For real momenta these singularities occur when,

$$t_{123}^2 + (s_{34} - s_{56} - s_{12}) t_{123} + s_{56} s_{12} = 0. \tag{3.9}$$

At the coplanar singularity the link between the box coefficients and the one-mass triangles implies that the latter also have coefficients with quadratic singularities. However the cancellation of the spurious singularities extends beyond these functions. At this singularity the dilogarithms within the two mass hard function simplify since $s_{12} = \alpha\beta s_{34}$, $s_{56} = (1 + \alpha)(1 + \beta)s_{34}$ and $t_{123} = \alpha(1 + \beta)s_{34}$, leading to,

$$I_4^{2mh} \sim \left(\text{Li}_2 \left(1 - \frac{\alpha}{(1 + \alpha)} \right) + \text{Li}_2 \left(1 - \frac{1 + \beta}{\beta} \right) \right). \tag{3.10}$$

These must cancel against another integral function containing dilogarithms: the three-mass triangle being the only possibility. The cancellation of spurious singularities can thus be expressed as,

$$[c_i I_4^i + d_i I_3^{1m\ i} + d^{3m} I_3^{3m}]_{[a|K|b]=0} = \text{finite}. \tag{3.11}$$

This imposes a significant constraint on the three-mass triangle coefficient. One approach is to change basis to one where this cancellation is automatic. This process is essentially the same as that of ref. [8, 27] where the three-mass triangle functions arise in $e^+ + e^- \rightarrow$ four parton scattering. We can generate this combination using the identity (3.5),

$$I_4^{D=4} - \frac{1}{2N_4} \left[\sum_i \alpha_i \gamma_i I_3^{(i)} \right] = (-1 + 2\epsilon) \frac{\hat{\Delta}_4}{2N_4} I_4^{D=6}. \tag{3.12}$$

For the two-mass hard box we have,

$$\frac{\hat{\Delta}_4}{2N_4} = -2 \left(\frac{\text{tr}(k_{i-1} \not{P} k_i \not{P})}{ST^2} \right) = -2 \frac{[i|P|i-1][i-1|P|i]}{s_{i-1i}(P^2)^2}. \tag{3.13}$$

Thus $\hat{\Delta}_4/2N_4 \rightarrow 0$ at the coplanar singularity and since $I_4^{D=6}$ is finite at this (unphysical) singularity we must have,

$$I_4^{D=4} - \frac{1}{2N_4} \left[\sum_i \alpha_i \gamma_i I_3^{(i)} \right] \rightarrow 0, \tag{3.14}$$

at the coplanar singularity. Up to a scaling factor this is precisely the L_{S1} function of ref. [8]. As this combination includes a three-mass triangle, the coefficient of this three-mass triangle in the $D = 4$ basis must contain a term,

$$-\frac{\alpha_i \gamma_i}{2N_4} c_4^{2mh}, \tag{3.15}$$

¹For real momenta $[4|K|3]$ and $[3|K|4]$ vanish simultaneously. However, by continuing to complex momenta we can find a point where only one of them vanishes, e.g. if $k_1 + k_2 \sim \lambda_x \bar{\lambda}_3 + \lambda_4 \bar{\lambda}_y$, $[3|K|4] = 0$ but $[4|K|3] \neq 0$.

which suggests the term,

$$\frac{-[2|K_{123}|4][2|K_{123}|5][3|K_{123}|5]}{[1|K_{123}|4][3|K_{123}|4]^2[3|K_{123}|6]t_{123}} \frac{[2|K_{123}|5](2s_{12}s_{56} - (s_{12} + s_{56} - s_{34})t_{123})}{2\langle 56 \rangle [12]}, \quad (3.16)$$

within d^{3m} . Since the three-mass triangle is the “daughter” of three different two-mass-hard boxes, each with a different quadratic singularity, we have three such terms in the coefficient.

This expression gives an amplitude from which the quadratic spurious singularity is absent. However we have introduced a further fictitious singularity: a t_{123} pole. The three-mass triangle should not contain such a pole. We can “fix” this by adding an extra term, giving,

$$\frac{-[2|K_{123}|4][2|K_{123}|5][3|K_{123}|5]}{[1|K_{123}|4][3|K_{123}|4]^2[3|K_{123}|6]t_{123}} \left(\frac{[2|K_{123}|5](2s_{12}s_{56} - (s_{12} + s_{56} - s_{34})t_{123})}{2\langle 56 \rangle [12][3|K_{123}|4]} + \langle 13 \rangle [64] \right). \quad (3.17)$$

This process fixes the leading quadratic coplanar pole. Fixing the remaining linear singularity gives more terms in the amplitude. Repeating the process as before we can deduce that,

$$\frac{1}{[3|K|4]^2} \left(I_4^{D=4} - \frac{1}{2N_4} \left[\sum_i \alpha_i \gamma_i I_3^{(i)} \right] + \frac{\Delta_4}{(2N_4)^2} \left[\sum_i \alpha_i \gamma_i I_3^{D=6,(i)} \right] \right) \rightarrow \text{finite}, \quad (3.18)$$

at the coplanar singularity. The function,

$$J_4 \equiv \left(I_4^{D=4} - \frac{1}{2N_4} \left[\sum_i \alpha_i \gamma_i I_3^{(i)} \right] + \frac{\Delta_4}{(2N_4)^2} \left[\sum_i \alpha_i \gamma_i I_3^{D=6,(i)} \right] \right), \quad (3.19)$$

is a combination of the $D = 4$ two-mass box, $D = 4$ triangle functions and $D = 4$ bubble functions. If we took the box coefficients and used the J_4 functions as a basis rather than the I_4 functions, we could extend the box contributions,

$$\sum_i c_i I_4^i \rightarrow \sum_i c_i J_4^i, \quad (3.20)$$

to obtain an expression containing much of the three-mass triangle and bubble contributions to the amplitude. This would be an expression without $[3|K_{123}|4]$, $[1|K_{561}|2]$ or $[5|K_{345}|6]$ singularities. It may however contain linear singularities due to $[5|K_{561}|2]$, $[1|K_{456}|4]$ or $[3|K_{123}|6]$ vanishing.

Looking carefully at the full singularity structure, after some trial and error, we are led to the expression for the three-mass triangle coefficient,

$$d_{3m}^{\{\{1^-2^+\},\{3^-4^+\},\{5^-6^+\}\}} \times (-i) = \quad (3.21)$$

$$\frac{[4|K_{345}|1][5|K_{345}|1][4|K_{345}|6]}{[5|K_{345}|2][3|K_{345}|6][5|K_{345}|6]t_{345}} \left(\frac{[4|K_{345}|1](2s_{12}s_{34} + (s_{56} - s_{12} - s_{34})t_{345})}{2\langle 12 \rangle [34][5|K_{345}|6]} + \langle 35 \rangle [26] \right)$$

$$\frac{[6|K_{561}|2][1|K_{561}|3][6|K_{561}|3]}{[1|K_{561}|2][5|K_{561}|2][1|K_{561}|4]t_{561}} \left(\frac{[6|K_{561}|3](2s_{34}s_{56} + (s_{12} - s_{34} - s_{56})t_{561})}{2\langle 34 \rangle [56][1|K_{561}|2]} + \langle 51 \rangle [42] \right)$$

$$\frac{[2|K_{123}|4][2|K_{123}|5][3|K_{123}|5]}{[1|K_{123}|4][3|K_{123}|4][3|K_{123}|6]t_{123}} \left(\frac{[2|K_{123}|5](2s_{12}s_{56} - (s_{12} + s_{56} - s_{34})t_{123})}{2\langle 56 \rangle [12][3|K_{123}|4]} + \langle 13 \rangle [64] \right)$$

$$\begin{aligned}
 & - \left(\frac{\langle 13 \rangle \langle 35 \rangle [26] [34] + \langle 13 \rangle \langle 15 \rangle [12] [46] + \langle 15 \rangle \langle 35 \rangle [24] [56]}{\Delta_3} \right) \times \\
 & \left(\frac{[5|K_{345}|1][4|K_{345}|6](t_{345}-t_{346})}{[5|K_{345}|2][3|K_{345}|6][5|K_{345}|6]} + \frac{[6|K_{561}|2][1|K_{561}|3](t_{561}-t_{562})}{[1|K_{561}|2][5|K_{561}|2][1|K_{561}|4]} + \frac{[2|K_{123}|4][3|K_{123}|5](t_{123}-t_{124})}{[1|K_{123}|4][3|K_{123}|4][3|K_{123}|6]} \right. \\
 & \left. - 2 \frac{[6|K_{234}|2][2|K_{456}|4][4|K_{612}|6] - [5|K_{345}|1][1|K_{561}|3][3|K_{123}|5]}{[5|K_{561}|2][1|K_{456}|4][3|K_{456}|6]} \right).
 \end{aligned}$$

We have confirmed this expression by comparison with a numerical evaluation of the triple cut. This provides an alternative form for the coefficient previously obtained in ref. [9]. Our form is free of irrational expressions and has a more manifest singularity structure.

We also obtain a rational form for the other six-point three-mass triangle coefficient,

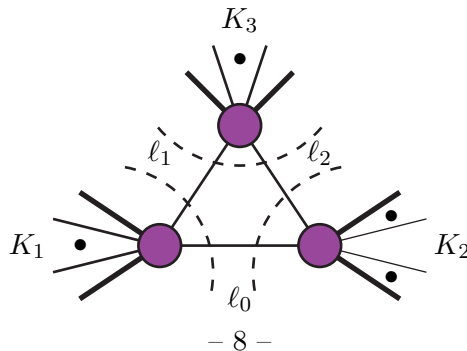
$$\begin{aligned}
 d_{3m}^{[\{2^-3^+\},\{4^-5^+\},\{6^{+1-}\}]} \times (-i) = & \tag{3.22} \\
 & \frac{[26][2|K_{345}|4][6|K_{345}|4]}{[12][2|K_{345}|3][2|K_{345}|5]t_{345}} \left(\frac{[6|K_{345}|4](2s_{61}s_{45} + (s_{23} - s_{61} - s_{45})t_{345})}{2[61]\langle 45 \rangle [2|K_{345}|3]} + \langle 12 \rangle [35] \right) \\
 & + \frac{\langle 51 \rangle [3|K_{234}|5][3|K_{234}|1]}{[56][4|K_{234}|5][2|K_{234}|5]t_{234}} \left(\frac{[3|K_{234}|1](2s_{23}s_{61} + (s_{45} - s_{23} - s_{61})t_{234})}{2[23]\langle 16 \rangle [4|K_{234}|5]} + \langle 24 \rangle [65] \right) \\
 & + \frac{[13]\langle 64 \rangle [3|K_{123}|4]}{[12]\langle 56 \rangle [1|K_{123}|6]t_{123}} \left(\frac{[3|K_{123}|4](2s_{45}s_{23} + (s_{61} - s_{45} - s_{23})t_{123})}{2[23]\langle 45 \rangle [1|K_{123}|6]} + \langle 12 \rangle [65] \right) \\
 & - \left(\frac{\langle 42 \rangle \langle 21 \rangle [32][65] + \langle 41 \rangle \langle 21 \rangle [61][35] + \langle 42 \rangle \langle 41 \rangle [45][36]}{\Delta_3} \right) \\
 & \times \left(2 \frac{[26]\langle 65 \rangle [36]\langle 64 \rangle - \langle 51 \rangle [12]\langle 41 \rangle [13]}{[12]\langle 56 \rangle [2|K_{561}|5]} + \frac{[13]\langle 46 \rangle (t_{123} - t_{623})}{\langle 56 \rangle [12][1|K_{123}|6]} \right. \\
 & \left. + \frac{[26][2|K_{345}|4](t_{345} - t_{245})}{[12][2|K_{345}|5][2|K_{345}|3]} + \frac{\langle 51 \rangle [3|K_{234}|5](t_{234} - t_{235})}{\langle 56 \rangle [2|K_{234}|5][4|K_{234}|5]} \right),
 \end{aligned}$$

which we have again confirmed by comparison with a numerical evaluation of the triple cut.

4. Analytic evaluation of the three-mass triangle coefficients

In this section we explore and refine some recent suggestions for using the analytic structure of triple cuts [21, 20, 28] to evaluate the three-mass triangle coefficients.

Consider a triple cut in an amplitude where all three corners are massive,



$$C_3 = \sum_{h_i \in S'} \int d^4 \ell_1 \delta(\ell_0^2) \delta(\ell_1^2) \delta(\ell_2^2) A_1 \left((\ell_0)^{h_1}, i_m, \dots, i_j, (-\ell_1)^{-h_2} \right) \quad (4.1)$$

$$\times A_2 \left((\ell_1)^{h_2}, i_{j+1}, \dots, i_l, (-\ell_2)^{-h_3} \right) \times A_3 \left((\ell_2)^{h_3}, i_{l+1}, \dots, i_{m-1}, (-\ell_0)^{-h_1} \right),$$

where the summation is over all possible intermediate states. As the momentum invariants, $K_m = k_{i_m} + k_{i_{m+1}} + \dots + k_{i_j}$ etc, are all non-null, there exist kinematic regimes in which the integration has non-vanishing support for real loop momentum. If we expand the amplitude in terms of a basis of integral functions (1.1), the only integral functions contributing to the triple cut are box functions and the specific three-mass triangle for the cut,

$$C_3 = \sum_i c_i (I_4^i)_{\text{triple-cut}} + d_{3m} (I_3^{3m})_{\text{triple-cut}}$$

$$= \sum_i \frac{c_i}{\kappa_i} + d_{3m} \frac{\pi}{2\sqrt{-\Delta_3}}, \quad (4.2)$$

For the two-mass hard and three-mass boxes that arise in the six and seven-point examples we discuss, the cuts of the box integral functions are,

$$\frac{1}{\kappa^{2mh}} = \pm \frac{\pi}{2(k_1 + k_2)^2 (k_2 + K_3)^2}, \quad \frac{1}{\kappa^{3m}} = \pm \frac{\pi}{2\left((k_1 + K_2)^2 (K_2 + K_3)^2 - K_2^2 K_4^2\right)}. \quad (4.3)$$

We will discuss the overall sign below.

Alternatively we can perform the cut integral (4.1). The triple cut is a one-parameter integral which can be calculated using algebraic methods [21]. We review the procedure for the general triple cut emphasising the geometric interpretation in the three-mass case as a contour integral.

The first step is to find a suitable parameterization of the cut momenta which satisfy $l_i^2 = 0$ with $l_1 = l_0 - K_1$ and $l_2 = l_0 + K_2$. As $\sum K_i = 0$, the momenta K_i define a plane. Within this plane there exists a momentum, a_0^μ , satisfying $a_0^2 = (a_0 - K_1)^2 = (a_0 + K_2)^2$. Explicitly [20],

$$a_0^\mu = \frac{K_2^2}{2} \frac{K_1 \cdot K_2 + K_1^2}{K_1^2 K_2^2 - (K_1 \cdot K_2)^2} K_1^\mu - \frac{K_1^2}{2} \frac{K_1 \cdot K_2 + K_2^2}{K_1^2 K_2^2 - (K_1 \cdot K_2)^2} K_2^\mu. \quad (4.4)$$

In the three-mass case, $|a_0| \neq 0$, the cut momenta are real and for a_0 time-like can be parameterised in the form,

$$\ell_0^\mu = a_0^\mu + \rho(\cos \theta m^\mu + \sin \theta n^\mu), \quad (4.5)$$

where $\rho = \sqrt{-a_0^2}$ and $0 \leq \theta \leq 2\pi$. The vectors m and n are mutually orthogonal unit vectors which are orthogonal to the (K_1, K_2) plane; $(m \cdot n) = (m \cdot K_i) = (n \cdot K_i) = 0$. For a_0 space-like, a hyperbolic parameterization can be used. If we now define the complex null momenta, $r = \frac{\rho}{2}(m + in)$ and $\bar{r} = \frac{\rho}{2}(m - in)$, we recover the parameterization used in [20, 21],²

$$\ell_i^\mu = t \bar{r}^\mu + \frac{1}{t} r^\mu + a_i^\mu, \quad (4.6)$$

²The t in our parameterization and that in [21] are related by a scaling

where $t = e^{i\theta}$, $a_1 = a_0 - K_1$ and $a_2 = a_0 + K_2$.

We can define null momenta \hat{K}_i in the plane of the K_i via [21],

$$\begin{aligned}\hat{K}_1 &= \frac{\gamma^2}{\gamma^2 - K_1^2 K_2^2} \left(K_1 - \frac{K_1^2}{\gamma} K_2 \right), \\ \hat{K}_2 &= \frac{\gamma^2}{\gamma^2 - K_1^2 K_2^2} \left(K_2 - \frac{K_2^2}{\gamma} K_1 \right),\end{aligned}\tag{4.7}$$

where $\gamma = K_1 \cdot K_2 + \frac{1}{2}\sqrt{-\Delta_3}$. In terms of the \hat{K}_i ,

$$r \sim \lambda_{K_2} \bar{\lambda}_{K_1}, \quad \bar{r} \sim \lambda_{K_1} \bar{\lambda}_{K_2}.\tag{4.8}$$

(We will drop the “hat” on K_i when it is clear from context that we are referring to the null form.)

For the spinors this parameterisation corresponds to,

$$\lambda_{l_i} = t \lambda_{K_1} + \alpha_{01} \lambda_{K_2}, \quad \bar{\lambda}_{l_i} = \frac{1}{t} (t \bar{\lambda}_{K_2} + \alpha_{02} \bar{\lambda}_{K_1}),\tag{4.9}$$

where,

$$\alpha_{01} = \frac{K_1^2(\gamma - K_2^2)}{\gamma^2 - K_1^2 K_2^2}, \quad \alpha_{02} = \frac{K_2^2(\gamma - K_1^2)}{\gamma^2 - K_1^2 K_2^2}.\tag{4.10}$$

With the parameterization (4.6) it is clear that the cut integration becomes a contour integration over the complex variable t with the contour being the unit circle. The integral then becomes,

$$\int d^4l \prod_i \delta(\ell_i^2)(\bullet) \longrightarrow \int dt J_t(\bullet),\tag{4.11}$$

where $J_t = 1/(4t\sqrt{\Delta_3})$ is the Jacobian. Regarding t as complex allows the integral to be performed analytically using contour methods. In the three-mass case the contour is well specified.

Parameterising the loop momenta according to (4.6) the product of tree amplitudes $A_1 A_2 A_3$ is a rational function of t . This rational function will have simple poles at $t = t_i \neq 0$ and, possibly, non-simple poles at $t = 0$. Poles in this product at $t = t_i \neq 0$ arise when one of the tree amplitudes factorises and some momentum, $\hat{P}(t)$, becomes null:

$$A \xrightarrow{\hat{P}^2 \rightarrow 0} \hat{A}_L \frac{1}{\hat{P}(t)^2} \hat{A}_R,\tag{4.12}$$

where \hat{A}_L and \hat{A}_R are tree amplitudes evaluated at the momenta where \hat{P}^2 vanishes. In general $\hat{P}^2 = 0$ gives two poles. For the six and seven-point examples we discuss, one of these poles gives the box contribution while the other gives no contribution. In these examples each box has at least one massless corner and we have $\hat{P} = l \pm a$, where a is the external momentum of a massless corner. Poles arise when either $\langle l a \rangle = 0$ or $[l a] = 0$. The original tree amplitudes will only contain one of these poles, so only one of the poles

can contribute to the triple cut. If the appropriate pole is inside the contour of integration, the contribution to the triple cut is of the form,

$$2\pi i \operatorname{Res} \left(\frac{A_1 A_2 \hat{A}_{3L} \hat{A}_{3R}}{4t\sqrt{\Delta_3} \hat{P}^2} \right) \Big|_{t=t_i}. \quad (4.13)$$

By comparison with the quadruple cut procedure, we see that the product of on-shell tree amplitudes reproduces the box coefficient up to a factor of 2. It is useful to compare the rest of this expression to the triple cut of the corresponding scalar box,

$$\int dt J_t \frac{1}{(l_0 - P)^2} = \int \frac{dt}{4t\sqrt{\Delta_3}} \frac{1}{\hat{P}^2}. \quad (4.14)$$

This has poles in identical positions, but both could in principle contribute. Denoting the two t -values for which $(l_0 - P)^2$ vanishes by t_{\pm} , we have,

$$\frac{1}{t(l_0 - P)^2} = \frac{-1}{(2\bar{r} \cdot P)(t - t_+)(t - t_-)} = \frac{-1}{(2\bar{r} \cdot P)(t_+ - t_-)} \left(\frac{1}{t - t_+} - \frac{1}{t - t_-} \right). \quad (4.15)$$

The two poles thus have equal but opposite residues. In the three-mass case, t_+ and t_- are the roots the quadratic equation,

$$2\bar{r} \cdot P t^2 + (2a_0 \cdot P - P^2)t + 2r \cdot P = 0, \quad (4.16)$$

so the product of the roots is,

$$t_+ t_- = \frac{r \cdot P}{\bar{r} \cdot P} \rightarrow |t_+ t_-| = 1. \quad (4.17)$$

One pole is always inside the unit circle and one is outside. Thus the triple cut of the box function always gives a contribution, but the sign depends on the kinematic point. In contrast, the original triple cut integral only receives contributions if the appropriate pole is inside the contour of integration.

In general $(A_1 A_2 A_3)/t$ can also have a pole at $t = 0$ and we denote the residue of this pole by ρ_0 . Using both approaches to evaluate the triple cut integral we then have,

$$C_3 = \sum_i 2\Theta(1 - |t_i|) \frac{c_i}{\tau_i} + \frac{2\pi\rho_0}{\sqrt{-\Delta_3}} = \sum_i -(-1)^{\Theta(1-|t_i|)} \frac{c_i}{\tau_i} + d_{3m} \frac{\pi}{2\sqrt{-\Delta_3}}, \quad (4.18)$$

where, $\tau_i = -(-1)^{\Theta(1-|t_i|)} \kappa_i$. We can rearrange this to give an expression free from Θ functions,

$$\frac{2\pi\rho_0}{\sqrt{-\Delta_3}} = \sum_i -\frac{c_i}{\tau_i} + d_{3m} \frac{\pi}{2\sqrt{-\Delta_3}} \equiv \mathcal{S} \equiv \mathcal{S}^{\text{box}} + \mathcal{S}^{\text{triangle}}, \quad (4.19)$$

which relates ρ_0 to a specific sum of box and triangle contributions. The box contributions are readily calculated either by quadruple cuts or by using the fact that they are half of the $t \neq 0$ residues. The latter approach provides a realisation of the quadruple cut procedure that is amenable to automation [21]. The three-mass triangle contribution to ρ_0 can thus readily be identified. A slightly different formulation involving integration over two

different regions (corresponding to the interior and exterior of the unit circle in this case) was presented in [21].

For any n -point NMHV amplitude the three tree amplitudes in the triple-cut of a three-mass triangle are of MHV type. When we parameterise the integral by t , each tree amplitude has a t^{-1} factor since each contains a $\langle l_i l_{i+1} \rangle^{-1}$ factor and, for example,

$$\begin{aligned} \langle l_0 l_1 \rangle &= \frac{[r|\not{l}_0\not{l}_1|r]}{[r l_0] [l_1 r]} = \frac{[r|(t\bar{r} + t^{-1}r + a_0)(t\bar{r} + t^{-1}r + a_0 - K_1)|r]}{[r l_0] [l_1 r]} \\ &= \frac{[r|t\bar{r}(a_0 - K_1) + ta_0\bar{r} + a_0(a_0 - K_1)|r]}{[r l_0] [l_1 r]} \\ &= t \frac{[r|\bar{r}(a_0 - K_1) + a_0\bar{r}|r]}{[r l_0] [l_1 r]} = t \frac{-[r|\bar{r}K_1|r]}{[r l_0] [l_1 r]}, \end{aligned} \quad (4.20)$$

using the orthogonality properties of r , \bar{r} , a_0 and K_1 . The $[r l_i]$ factors cancel overall as the product of tree amplitudes has no spinor weight in l_i . Thus, for each particle circulating in the loop, the integrand has a t^{-3} factor and ρ_0 must be extracted by expanding around this triple pole.

For the $\mathcal{N} = 1$ coefficients we present, summing over the particle types leads to cancellations. Relative to the case of a scalar in the loop, the $\mathcal{N} = 1$ multiplet has an overall factor. Denoting the three negative helicity external legs by m_i , this factor is,

$$\frac{(\langle l_0 m_1 \rangle \langle l_1 m_2 \rangle \langle l_2 m_3 \rangle - \langle l_2 m_1 \rangle \langle l_0 m_2 \rangle \langle l_1 m_3 \rangle)^2}{\langle l_0 m_1 \rangle \langle l_1 m_2 \rangle \langle l_2 m_3 \rangle \langle l_2 m_1 \rangle \langle l_0 m_2 \rangle \langle l_1 m_3 \rangle}. \quad (4.21)$$

Cancellations in the numerator give this expression an overall factor of t^2 implying that the full $\mathcal{N} = 1$ integrand diverges as t^{-1} . ρ_0 can then be extracted by taking a derivative:

$$\rho_0 \sim \frac{d}{dt} \left(t(A_1 A_2 A_3) \right)_{t \rightarrow 0}. \quad (4.22)$$

For $\mathcal{N} = 4$ the overall factor is that of $\mathcal{N} = 1$ squared and thus introduces a t^4 factor. For these amplitudes it is thus trivial to see that $\rho_0 = 0$ and there are no three-mass triangles present in the expansion. This argument easily extends to show that there are no three-mass triangles present in $\mathcal{N} = 8$ supergravity [29].

5. Canonical forms

We can use the techniques of the previous section to derive canonical forms for evaluating the coefficients of three mass triangles from the triple cut. In general we wish to expand the product of tree amplitudes in the triple cut as a sum of standard forms. Let us take as the starting point a term of the form,

$$\frac{\langle b l_0 \rangle}{\langle a l_0 \rangle}, \quad (5.1)$$

and let us carry out the parameterisation of l_0 given in (4.9) including a factor of t^{-1} from the measure to obtain the following integrand,

$$\frac{(t \langle K_1 b \rangle + \alpha_{01} \langle K_2 b \rangle)}{t(t \langle K_1 a \rangle + \alpha_{01} \langle K_2 a \rangle)} = \frac{1 \langle K_2 b \rangle}{t \langle K_2 a \rangle} + \frac{(\langle K_2 a \rangle \langle K_1 b \rangle - \langle K_1 a \rangle \langle K_2 b \rangle)}{\langle K_2 a \rangle (t \langle K_1 a \rangle + \alpha_{01} \langle K_2 a \rangle)}, \quad (5.2)$$

provided that $\langle a K_i \rangle \neq 0$. The contribution to the three-mass triangle is the residue at $t = 0$ minus half the residue at $t \neq 0$, namely,

$$\begin{aligned} \frac{\langle K_2 b \rangle}{\langle K_2 a \rangle} - \frac{(\langle K_2 a \rangle \langle K_1 b \rangle - \langle K_1 a \rangle \langle K_2 b \rangle)}{2 \langle K_2 a \rangle \langle K_1 a \rangle} & \quad (5.3) \\ &= \frac{(\langle K_2 a \rangle \langle K_1 b \rangle + \langle K_1 a \rangle \langle K_2 b \rangle)}{2 \langle K_2 a \rangle \langle K_1 a \rangle} \\ &= \frac{\langle a | (\hat{K}_1 \hat{K}_2 - \hat{K}_2 \hat{K}_1) | b \rangle}{2 \langle a | \hat{K}_1 \hat{K}_2 | a \rangle} = \frac{\langle a | (K_1 K_2 - K_2 K_1) | b \rangle}{2 \langle a | K_1 K_2 | a \rangle}, \end{aligned}$$

with the \hat{K}_i as defined as in eq. (4.12). When $\langle a K_1 \rangle = 0$ (i.e. $\lambda_a \sim \lambda_{K_1}$) there is no $t \neq 0$ pole and we have,

$$\frac{\langle \ell_0 b \rangle}{\langle \ell_0 K_1 \rangle} \longrightarrow \frac{\langle K_2 b \rangle}{\langle K_2 K_1 \rangle}. \quad (5.4)$$

Next, consider expressions of the form,

$$\frac{\langle a \ell_0 \rangle \langle b \ell_0 \rangle}{\langle \ell_0 | K_1 K_2 | \ell_0 \rangle}, \quad (5.5)$$

which are evaluated by replacing the K_i by \hat{K}_i ,

$$\begin{aligned} \frac{\langle a \ell_0 \rangle \langle b \ell_0 \rangle}{\langle \ell_0 | K_1 K_2 | \ell_0 \rangle} &= \frac{1}{(1 - K_1^2 K_2^2 / \gamma^2)} \frac{\langle a \ell_0 \rangle \langle b \ell_0 \rangle}{\langle \ell_0 | \hat{K}_1 \hat{K}_2 | \ell_0 \rangle} = \frac{1}{(1 - K_1^2 K_2^2 / \gamma^2)} \frac{\langle a \ell_0 \rangle \langle b \ell_0 \rangle}{\langle \ell_0 K_1 | [K_1 K_2] \langle K_2 \ell_0 \rangle}, \\ &= \frac{\langle a \ell_0 \rangle}{[K_1 K_2] (1 - K_1^2 K_2^2 / \gamma^2)} \left(\frac{\langle b K_1 \rangle}{\langle \ell_0 K_1 \rangle \langle K_2 K_1 \rangle} + \frac{\langle b K_2 \rangle}{\langle K_2 K_1 \rangle \langle K_2 \ell_0 \rangle} \right), \end{aligned} \quad (5.6)$$

which is two terms of the form (5.4). After some algebra, we can combine these terms to obtain,

$$\frac{\langle \ell_0 a \rangle \langle \ell_0 b \rangle}{\langle \ell_0 | K_1 K_2 | \ell_0 \rangle} \xrightarrow{\text{triangle}} \frac{\langle a | (K_1 K_2 - K_2 K_1) | b \rangle}{\Delta_3} = \frac{\langle a | [K_1, K_2] | b \rangle}{\Delta_3}. \quad (5.7)$$

We can extend this to,

$$\begin{aligned} \frac{\langle \ell_0 a \rangle \langle \ell_0 b \rangle \langle \ell_0 c \rangle}{\langle \ell_0 | K_1 K_2 | \ell_0 \rangle \langle \ell_0 d \rangle} &\xrightarrow{\text{triangle}} \frac{\langle b | [K_1, K_2] | d \rangle \langle c | [K_1, K_2] | a \rangle - \Delta_3 \langle b d \rangle \langle c a \rangle}{2 \Delta_3 \langle d | K_1 K_2 | d \rangle} \\ &+ \frac{\langle d b \rangle \langle d c \rangle \langle a | [K_1, K_2] | d \rangle}{2 \langle d | K_1 K_2 | d \rangle^2}. \end{aligned} \quad (5.8)$$

This result will be sufficient to obtain the three-mass triangle coefficients for the n -point NMHV $\mathcal{N} = 1$ contribution.

For a general $\mathcal{N} = 1$ amplitude we also need,

$$\begin{aligned} \frac{[A | \ell_0 | b \rangle \langle \ell_0 c \rangle}{\langle \ell_0 d \rangle} &\xrightarrow{\text{triangle}} - \frac{[A | a_0 | d \rangle (\langle d | [K_1, K_2] | b \rangle \langle d | [K_1, K_2] | c \rangle - \Delta_3 \langle d b \rangle \langle d c \rangle)}{8 \langle d | K_1 K_2 | d \rangle^2} \\ &+ \frac{[A | a_0 | b \rangle \langle d | [K_1, K_2] | c \rangle + [A | a_0 | c \rangle \langle d | [K_1, K_2] | b \rangle}{4 \langle d | K_1 K_2 | d \rangle}. \end{aligned} \quad (5.9)$$

6. n -point NMHV $\mathcal{N} = 1$ results

We consider an n -point amplitude with three negative helicity legs m_i . The triple cut vanishes unless there is precisely one external negative helicity leg at each corner. The product of the tree amplitudes will be,

$$\begin{aligned}
 & \sum_{h=0,\pm 1/2} A(\ell_0^h, r+1^+, \dots, m_1^-, \dots, s^+, -\ell_1^{-h}) \times A(\ell_1^h, s+1^+, \dots, m_2^-, \dots, t^+, -\ell_2^{-h}) \\
 & \quad \times A(\ell_2^h, t+1^+, \dots, m_3^-, \dots, r^+, -\ell_0^{-h}) \\
 & = A(\ell_0^0, r+1^+, \dots, m_1^-, \dots, s^+, -\ell_1^0) \times A(\ell_1^0, s+1^+, \dots, m_2^-, \dots, t^+, -\ell_2^0) \\
 & \quad \times A(\ell_2^0, t+1^+, \dots, m_3^-, \dots, r^+, -\ell_0^0) \times \rho.
 \end{aligned} \tag{6.1}$$

The ρ -factor arises from summing over the multiplet and is,

$$\begin{aligned}
 \rho & = \frac{(\langle m_1 \ell_0 \rangle \langle m_2 \ell_1 \rangle \langle m_3 \ell_2 \rangle - \langle m_1 \ell_1 \rangle \langle m_2 \ell_2 \rangle \langle m_3 \ell_0 \rangle)^2}{\langle m_1 \ell_0 \rangle \langle m_2 \ell_1 \rangle \langle m_3 \ell_2 \rangle \langle m_1 \ell_1 \rangle \langle m_2 \ell_2 \rangle \langle m_3 \ell_0 \rangle} \\
 & = \frac{\langle X \ell_0 \rangle^2}{\langle m_1 \ell_0 \rangle \langle m_2 \ell_1 \rangle \langle m_3 \ell_2 \rangle \langle m_1 \ell_1 \rangle \langle m_2 \ell_2 \rangle \langle m_3 \ell_0 \rangle [l_1 l_2]^2},
 \end{aligned} \tag{6.2}$$

where,

$$\langle X \rangle = |m_1 \rangle \langle \langle m_3 | K_3 K_1 | m_2 \rangle + \langle m_2 m_3 \rangle (K_2^2 - K_1^2) + |m_3 \rangle \langle m_1 | K_1 K_3 | m_2 \rangle. \tag{6.3}$$

The cut is of the form,

$$\begin{aligned}
 & \frac{1}{\prod_{i \neq r,s,t} \langle i i+1 \rangle} \times \frac{\langle m_1 \ell_0 \rangle \langle m_1 \ell_1 \rangle}{\langle \ell_0 r+1 \rangle \langle s \ell_1 \rangle \langle \ell_1 \ell_0 \rangle} \times \frac{\langle m_2 \ell_1 \rangle \langle m_2 \ell_2 \rangle}{\langle \ell_1 s+1 \rangle \langle t \ell_2 \rangle \langle \ell_2 \ell_1 \rangle} \\
 & \quad \times \frac{\langle m_3 \ell_2 \rangle \langle m_3 \ell_0 \rangle}{\langle \ell_2 t+1 \rangle \langle r \ell_0 \rangle \langle \ell_0 \ell_2 \rangle} \times \frac{\langle X \ell_0 \rangle^2}{[l_1 l_2]^2}.
 \end{aligned} \tag{6.4}$$

In this we can combine, $\langle \ell_1 \ell_0 \rangle [l_1 l_2] \langle \ell_0 \ell_2 \rangle = -\langle \ell_0 | K_1 K_2 | \ell_0 \rangle$ and $\langle \ell_2 \ell_1 \rangle [l_1 l_2] = -K_3^2$. We can write (6.4) in terms of just one of the cut momenta using identities of the form,

$$\frac{\langle \ell_1 b \rangle}{\langle \ell_1 a \rangle} = \frac{\langle \ell_0 \ell_2 \rangle [l_2 l_1] \langle \ell_1 b \rangle}{\langle \ell_0 \ell_2 \rangle [l_2 l_1] \langle \ell_1 a \rangle} = \frac{\langle \ell_0 | K_2 K_3 | b \rangle}{\langle \ell_0 | K_2 K_3 | a \rangle} \equiv \frac{\langle \ell_0 b^{32} \rangle}{\langle \ell_0 a^{32} \rangle}, \tag{6.5}$$

where we use the compact notation $|a^{ij} \rangle \equiv K_j K_i |a \rangle$. This gives,

$$\frac{1}{K_3^2 \prod_{i \neq r,s,t} \langle i i+1 \rangle} \times \frac{\langle m_1 \ell_0 \rangle \langle m_1^{32} \ell_0 \rangle \langle m_2^{32} \ell_0 \rangle \langle m_2^{31} \ell_0 \rangle \langle m_3 \ell_0 \rangle \langle m_3^{31} \ell_0 \rangle}{\prod_{y \in Y_6} \langle \ell_0 y \rangle} \times \frac{\langle X \ell_0 \rangle^2}{\langle \ell_0 | K_1 K_2 | \ell_0 \rangle}, \tag{6.6}$$

where $Y_6 = \{r, r+1, s^{32}, (s+1)^{32}, t^{31}, (t+1)^{31}\}$. Using partial fractions this can be written as,

$$\frac{1}{K_3^2 \prod_{i \neq r,s,t} \langle i i+1 \rangle} \times \sum_{y \in Y_6} \frac{\langle m_1^{32} y \rangle \langle m_2^{32} y \rangle \langle m_2^{31} y \rangle \langle m_3 y \rangle \langle m_3^{31} y \rangle}{\prod_{z \in Y_6, z \neq y} \langle z y \rangle} \times \frac{\langle m_1 \ell_0 \rangle \langle X \ell_0 \rangle^2}{\langle y \ell_0 \rangle \langle \ell_0 | K_1 K_2 | \ell_0 \rangle}. \tag{6.7}$$

This is simply a sum of canonical forms (5.8) and so the three-mass triangle coefficient is,

$$\frac{-i}{K_3^2 \prod_{i \neq r,s,t} \langle i i + 1 \rangle} \sum_{y \in Y_6} \frac{\langle m_1^{32} y \rangle \langle m_2^{32} y \rangle \langle m_3^{31} y \rangle \langle m_3 y \rangle \langle m_3^{31} y \rangle}{\prod_{z \in Y_6, z \neq y} \langle z y \rangle} \times \left(\frac{\langle m_1 | [K_1, K_2] | y \rangle \langle X | [K_1, K_2] | X \rangle}{2\Delta_3 \langle y | K_1 K_2 | y \rangle} + \frac{\langle y X \rangle \langle y m_1 \rangle \langle X | [K_1, K_2] | y \rangle}{2 \langle y | K_1 K_2 | y \rangle^2} \right). \quad (6.8)$$

7. Beyond NMHV and $\mathcal{N} = 0$

We can, in principle, use the methods described above to obtain the three-mass triangle coefficients for amplitudes beyond NMHV or with less supersymmetry, i.e. $\mathcal{N} = 0$. In this section we outline how this may be performed.

In general the product of the tree amplitudes will be a sum of terms which we treat individually. The first step is to turn each term into a function of a single loop momentum, say $\ell_0 \equiv \ell$. Furthermore we will make this a function depending predominantly on terms such as $\langle a \ell \rangle$ rather than $[a \ell]$. We can do this via replacements such as,

$$\begin{aligned} \frac{\langle \ell_1 b \rangle}{\langle \ell_1 a \rangle} &= \frac{\langle \ell_0 \ell_2 \rangle [\ell_2 \ell_1] \langle \ell_1 b \rangle}{\langle \ell_0 \ell_2 \rangle [\ell_2 \ell_1] \langle \ell_1 a \rangle} = \frac{\langle \ell | K_2 K_3 | b \rangle}{\langle \ell | K_2 K_3 | a \rangle}, \\ \frac{[\ell_1 b]}{[\ell_1 a]} &= \frac{\langle \ell | K_1 | b \rangle}{\langle \ell | K_1 | a \rangle}, \\ \frac{\langle \ell_0 b \rangle}{\langle \ell_0 a \rangle} &= \frac{\langle \ell_0 \ell_2 \rangle [\ell_2 \ell_1] \langle \ell_1 \ell_0 \rangle [\ell_0 b]}{\langle \ell_0 \ell_2 \rangle [\ell_2 \ell_1] \langle \ell_1 \ell_0 \rangle [\ell_0 a]} = \frac{\langle \ell | K_2 K_3 K_1 | b \rangle}{\langle \ell | K_2 K_3 K_1 | a \rangle}. \end{aligned} \quad (7.1)$$

While it is always possible to make the above replacements for the triple-cut, analogous replacements are not always possible for the two-particle cut. Carrying out all possible replacement as described above, each term in the cut can be written in the form,

$$\frac{\prod_{i=1}^n \langle A_i \ell \rangle}{\prod_{j=1}^n \langle B_j \ell \rangle \prod_{l=1}^p (\ell + Q_l)^2} \times \prod_{k=1}^q [C_k | \ell | D_k]. \quad (7.2)$$

We can tackle the massive propagators by utilising the identity,

$$\frac{1}{(\ell + Q)^2} [C \ell] = \frac{1}{(\ell + Q)^2} \frac{[C | K_1 (K_1 + Q) Q | \ell]}{\langle \ell | K_1 Q | \ell \rangle} - \frac{[C | K_1 | \ell]}{\langle \ell | K_1 Q | \ell \rangle}. \quad (7.3)$$

Using the parameterization (4.9) we see that the first and third terms are $\mathcal{O}(t^0)$ near $t = 0$, while the second is $\mathcal{O}(t^1)$.³ Multiple application of this identity leads to a sum of terms of the form,

$$\frac{\prod_{i=1}^{n+2p} \langle A_i \ell \rangle}{\prod_{j=1}^{n+2p} \langle B_j \ell \rangle} \left(\prod_{k=1}^{q-p} [C_k | \ell | D_k] \right), \quad (7.4)$$

³It is worth noting that this counting would not have held had the numerators contained $\langle \ell | K_1 K_2 | \ell \rangle$ rather than of $\langle \ell | K_1 Q | \ell \rangle$. Recalling that \hat{K}_1 and \hat{K}_2 are null vectors in the (K_1, K_2) plane, we see that in this case there are cancellations within the denominator which give it an extra overall factor of t .

together with terms that vanish at $t = 0$ - these only contribute to box coefficients and can be neglected.

In general the Yang-Mills amplitudes at each corner contribute an effective overall momentum power of ℓ^1 . Thus the $\mathcal{N} = 0$ amplitude contains terms with momentum power up to ℓ^3 . For $\mathcal{N} = 1$ contributions, summing over the multiplet cancels the two leading powers and we expect the three-point integrals to go as ℓ^1 and thus expect $q = p + 1$. In this case (7.4) is expressed as a sum of canonical terms of the form evaluated in (5.7)–(5.9). For general $\mathcal{N} = 0$ contributions we expect terms with $q = p + 3$ which could be obtained using higher power analogues of (5.9).

8. Conclusions

We have discussed a range of techniques that utilise the analytic properties of one-loop amplitudes to generate the coefficients of one-loop integral functions. By combining these carefully we have generated explicit expressions for the coefficients of the three-mass triangle functions in any NMHV n -point $\mathcal{N} = 1$ amplitude.

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